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Collinear Asymptotic Dynamics for Massive Particles. Reggeization and Eikonalization.*

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Abstract

The dynamics of massive particles in the collinear high momentum regime is investigated. Methods hitherto exploiting large time asymptotic Hamiltonians in the Dirac picture for the treatment of infrared divergences are adapted to the collinear asymptotic dynamics. The essential role of time ordering of the dressing operators is brought out.

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1 Introduction

For the treatment of infrared divergencies and the problem of the Coulomb phase it has been known for a long time that perturbative contributions up to infinite order have to be taken into account. Kinoshita [1], Lee and Nauenberg [2] showed that maintaining unitarity in the approximation is the key for solving this problem. Khulish and Faddeev [3] pointed out that both phenomena are dominated by the asymptotic dynamics of QED. They separated out that part of the Hamiltonian which is essential for large time scales.

The dynamics at large times is dominated by the interaction involving the long wave length photons. In the Dirac picture the temporal variation of their contribution to the interaction Hamiltonian is small. Actually, for the energy of the photons going to zero the corresponding interaction Hamiltonian density in momentum space becomes stationary.

Similarly, photons collinear to electrons lead to parts in the interaction density stationary in the limit of high momentum [4].

In the case of massive particles only in the high momentum limit the stationary regime is sustained.

Adapting the methods of asymptotic dynamics [5, 6, 7] to the massive case we shall show that they lead to Reggeization [8, 9, 10]. While in the treatment of infrared dynamics time ordering of the dressing operator did not come into play, here it will be of essential importance.

We shall also show that the other typical high energy approximation, i. e. eikonalization [11], can be obtained using our methods.

In this paper for developing the appropriate adaptation of our method we study the φ^3 theory, since it is the simplest model for which Reggeization has been demonstrated.

2 Approximate Unitarity

Calculations of bremsstrahlung processes require in a unitary approximation the inclusion of an infinity of orders in the coupling constant to ensure a finite total cross section.

One way of reaching this aim is through an approximate calculation of Dyson's time evolution operator $U(t,t_0)$ determined by the equation

$$\mathrm{i}\frac{\partial}{\partial t}U(t,t_0) = H_{\mathrm{I}}(t)U(t,t_0) \tag{2.1}$$

with the well known formal solution

$$U(t,t_0) = \mathrm{T}\exp\left\{-\mathrm{i}\int\limits_{t_0}^{t} H_{\mathrm{I}}(t')\,\mathrm{d}t'\right\} \quad , \tag{2.2}$$

where $H_{\mathrm{I}}(t)$ is the interaction Hamiltonian in the Dirac picture.

For definiteness we exemplify our method on the φ^3 model. The interaction Hamiltonian

$$H_{\mathrm{I}} = -\frac{g}{3!}\int :\varphi^3(x):\,\mathrm{d}^3x \tag{2.3}$$

can be written as

$$H_{\mathrm{I}} = -\frac{g}{3!}\int \left\{\varphi^{(-)3}(x) + \varphi^{(+)3}(x) + 3\varphi^{(-)2}(x)\varphi^{(+)}(x) + 3\varphi^{(-)}(x)\varphi^{(+)2}(x)\right\}\mathrm{d}^3x \tag{2.4}$$

by splitting the field operator $\varphi(x) = \varphi^{(-)}(x) + \varphi^{(+)}(x)$ into the creation and annihilation parts.

As in the argument of Kulish and Faddeev [3] in the case of bremsstrahlung and Coulomb phase we select those contributions to the interaction that are most important at large time scales. At first we neglect those terms of H_{I} that create or annihilate three particles and get

$$\tilde{H}_{\mathrm{I}} = H_{\mathrm{I}}^{(+)} + H_{\mathrm{I}}^{(-)} \tag{2.5}$$

with

$$H_{\mathrm{I}}^{(+)} = -\frac{g}{2} \int \varphi^{(-)2}(x)\varphi^{(+)}(x)\,\mathrm{d}^3x \quad , \quad H_{\mathrm{I}}^{(-)} = -\frac{g}{2} \int \varphi^{(-)}(x)\varphi^{(+)2}(x)\,\mathrm{d}^3x \quad . \tag{2.6}$$

or in momentum space

$$H_{\mathrm{I}}^{(+)} = -\frac{g}{4} \int \frac{\mathrm{d}\tilde{k}\,\mathrm{d}\tilde{p}}{E(\vec{p}-\vec{k})}\, a^\dagger(\vec{p}-\vec{k})a^\dagger(\vec{k})a(\vec{p}) \exp\left\{\mathrm{i}[E(\vec{k}) + E(\vec{p}-\vec{k}) - E(\vec{p})]t\right\} \tag{2.7}$$

where

$$\mathrm{d}\tilde{k} = \frac{\mathrm{d}^3k}{(2\pi)^3 2k^0} \quad . \tag{2.8}$$

Collinear asymptotic dynamics (for $p \gg m$, where $p = |\vec{p}|$) will be incorporated by restricting the $\vec{k}$-integration to a region $\Omega(\vec{p})$ in which $E(\vec{k}) + E(\vec{p}-\vec{k}) - E(\vec{p})$ tends to zero for $p \to \infty$. A convenient choice for $\Omega(\vec{p})$ is given by the requirement

$$k^\mu l_\mu \leq \sigma^2 \tag{2.9}$$

where l^μ is the momentum vector of the second created particle, $l^\mu = (E(\vec{p}-\vec{k}), \vec{p}-\vec{k})$, and σ is a scale parameter. For $\sigma > m$ the region $\Omega(\vec{p})$ defined in this way restricts $k^\parallel$ and $\vec{k}^\perp$ to an ellipsoid

$$\frac{(k^\parallel - \frac{p}{2})^2}{a^2} + \frac{\vec{k}^{\perp 2}}{b^2} \leq 1 \tag{2.10}$$

with the principal axes a and b given by

$$a^2 = \frac{p^2}{4}\left(\frac{\sigma^2 - m^2}{\sigma^2 + m^2}\right) + \frac{\sigma^2 - m^2}{2} \quad \text{and} \quad b^2 = \frac{\sigma^2 - m^2}{2} \quad . \tag{2.11}$$

For $p \gg \sigma \gg m$ we have

$$a \approx \frac{p}{2} \quad . \tag{2.12}$$

The calculation of $U(t, t_0)$ is greatly simplified if the time-ordered exponential is suitably factorized. This is facilitated by a time dependent similarity transformation. For two operators $A(t)$, $B(t)$, not necessary Hermitean, one has

$$\mathrm{T}\exp\left\{(-\mathrm{i})\int_{t_0}^{t} A(t')\,\mathrm{d}t' + (-\mathrm{i})\int_{t_0}^{t} B(t')\,\mathrm{d}t'\right\} = \tag{2.13}$$

$$= \quad \mathrm{T}\exp\left\{-\mathrm{i}\int_{t_0}^{t} A(t')\,\mathrm{d}t'\right\} \mathrm{T}\exp\left\{-\mathrm{i}\int_{t_0}^{t} W_1^{-1}(t', t_0)B(t')W_1(t', t_0)\,\mathrm{d}t'\right\}$$

with

$$W_1(t,t_0) = \mathrm{T}\exp\left\{-\mathrm{i}\int\limits_{t_0}^{t} A(t')\,\mathrm{d}t'\right\} \quad . \tag{2.14}$$

The validity of the factorization of the T-product follows from a factored ansatz $U = W_1 W$ for the time evolution operator leading to the differential equation

$$\mathrm{i}\frac{\partial}{\partial t}W(t,t_0) = W_1^{-1}(t,t_0)B(t)W_1(t,t_0)W(t,t_0) \tag{2.15}$$

having again as solution a time-ordered exponential

$$W(t,t_0) = \mathrm{T}\exp\left\{-\mathrm{i}\int\limits_{t_0}^{t} W_1^{-1}(t',t_0)B(t')W_1(t',t_0)\,\mathrm{d}t'\right\} \quad . \tag{2.16}$$

For further use we quote a formula already given by Dyson [12]

$$\begin{aligned} & W_1^{-1}(t',t_0)B(t')W_1(t',t_0) = \\ & = \sum_{n=0}^{\infty} \mathrm{i}^n \int\limits_{t_0}^{t} \mathrm{d}t_n \int\limits_{t_0}^{t_n} \mathrm{d}t_{n-1} \cdots \int\limits_{t_0}^{t_2} \mathrm{d}t_1 [A(t_1),[A(t_2),\ldots[A(t_n),B(t)]\ldots] \quad . \end{aligned} \tag{2.17}$$

In our context $A(t)$, $B(t)$ will be identified with $H_{\mathrm{I}}^{(+)}(t)$, $H_{\mathrm{I}}^{(-)}(t)$, respectively. We will use the truncated expression

$$W_1^{-1}BW_1 = B(t) + \mathrm{i}\int\limits_{t_0}^{t} [A(t_1),B(t)]\,\mathrm{d}t_1 + O(g^3) \quad . \tag{2.18}$$

Again applying eq. (2.13) we can introduce

$$W_2(t,t_0) = \mathrm{T}\exp\left\{-\mathrm{i}\int\limits_{t_0}^{t} B(t')\,\mathrm{d}t'\right\} \tag{2.19}$$

and

$$W_3(t,t_0) = \mathrm{T}\exp\left\{\int\limits_{t_0}^{t}\mathrm{d}t_2\int\limits_{t_0}^{t_2}\mathrm{d}t_1[A(t_1),B(t_2)]\right\} \tag{2.20}$$

(note that the time ordering in W_3 refers to t_2). In this way we get an approximate expression for the time evolution operator

$$U(t,t_0) = W_1(t,t_0)W_2(t,t_0)W_3(t,t_0) \quad . \tag{2.21}$$

It fulfills an approximate unitarity relation of the kind

$$U^\dagger(t,t_0)U(t,t_0) = \exp O(g^3) \quad . \tag{2.22}$$

A similar approach has been successfully used in bremsstrahlung calculations [4]. In what follows we are going to show that the above approximation produces Reggeization and eikonalization of two-body amplitudes for elastic scattering.

Let us give the explicit formula for W_3 using $A(t) = H_{\mathrm{I}}^{(+)}(t)$, $B(t) = H_{\mathrm{I}}^{(-)}(t)$:

$$\begin{aligned} W_3(t,t_0) &= \mathrm{T}\exp\left\{\frac{g^2}{4}\int\limits_{x^0=t_0}^{t} \mathrm{d}^4x \int\limits_{y^0=t_0}^{x^0} \mathrm{d}^4y \left(\varphi^{(-)2}(y)\varphi^{(+)2}(x)\Delta^+(y-x)\right.\right. \\ &\qquad \left.\left. -4\varphi^{(-)}(x)\varphi^{(-)}(y)\varphi^{(+)}(x)\varphi^{(+)}(y)\Delta^+(x-y)\right)\right\} \end{aligned} \tag{2.23}$$

with

$$\Delta^+(x-y) = [\varphi^{(+)}(x), \varphi^{(-)}(y)] \quad . \tag{2.24}$$

In the expression for W_3 we have neglected a part that corresponds to renormalization corrections which will not be considered here.

3 Reggeization

In the case of soft electromagnetic bremsstrahlung the asymptotic interaction Hamiltonian is linear in the photon field. The photon creating part of this Hamiltonian corresponding to our $H_{\mathrm{I}}^{(+)}(t)$ commutes with itself at different times, since the electromagnetic current can be well approximated by the classical one. Thus, the time-ordering of the exponential is superfluous.

In contradistinction, $H_{\mathrm{I}}^{(+)}(t)$, eq. (2.6), is a product of $\varphi^{(-)2}$ and $\varphi^{(+)}$ and therefore does not commute with itself at different times, so that the time-ordering in W_1 remains essential. A similar comment holds true for W_2.

Let us start with a suitable formula for the elastic amplitude $\mathcal{T}_{\mathrm{if}}(p_1+p_2 \to p_3+p_4)$:

$$\mathcal{T}_{\mathrm{if}} = T + T^{\mathrm{cr}} \quad , \tag{3.1}$$

where

$$T = -\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{e}^{-\mathrm{i}p_2y+\mathrm{i}p_4x}\Theta(x^0-y^0)\langle\vec{p}_3,\mathrm{out}|j_{\mathrm{H}}(x)j_{\mathrm{H}}(y)|\vec{p}_1,\mathrm{in}\rangle \tag{3.2}$$

and

$$T^{\mathrm{cr}} = T(p_2 \leftrightarrow -p_4) \quad . \tag{3.3}$$

In eq. (3.2) the Heisenberg current is denoted by the index "H".

The matrix element T can be written as

$$\begin{aligned} T &= -\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{e}^{-\mathrm{i}p_2y+\mathrm{i}p_4x}\Theta(x^0-y^0) \\ &\quad \times\langle\vec{p}_3|U^\dagger(x^0,-\infty)j(x)U(x^0,y^0)j(y)U(y^0,-\infty)|\vec{p}_1\rangle \quad . \end{aligned} \tag{3.4}$$

In what follows we shall consider small transfers $t=(p_1-p_3)^2$ at high energies. It will be convenient to work in the laboratory frame of the particle with momentum p_2; p_1, p_3 are chosen to be large momenta. Detailed calculations will be made in the interaction picture and we expect that the leading contribution to the reggeized amplitude is real and comes from the quasi-stable multiparticle configurations. Therefore it is reasonable to replace

$$U(x^0,y^0) \to \mathbf{1} \quad . \tag{3.5}$$

Inserting a complete set of states between $j(x)$ and $j(y)$ yields

$$T = (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4)\mathcal{M}_2 \quad , \tag{3.6}$$

with

$$\mathcal{M}_2 = (2\pi)^3 \sum_{n=1}^{\infty} \frac{1}{n!} V_n \quad , \tag{3.7}$$

where

$$V_n = \int \mathrm{d}\tilde{k}_1 \cdots \mathrm{d}\tilde{k}_n \, \frac{\delta(\vec{p}_1 + \vec{p}_2 - \sum_{i=1}^{n} \vec{k}_i)}{{p_1}^0 + {p_2}^0 - \sum_{i=1}^{n} {k_i}^0 + \mathrm{i}\varepsilon} \, G_n^*(\vec{p}_3) G_n(\vec{p}_1) \tag{3.8}$$

and

$$G_n(\vec{p}) = \langle \vec{k}_1, \ldots, \vec{k}_n | j(0) U(0, -\infty) | \vec{p} \rangle \quad . \tag{3.9}$$

As we are working in the laboratory frame of the particle with momentum p_2, the current $j(0)$ is connected with the interaction of this slow particle. Since we expect that $U(0, -\infty)$ alone accounts for the production of energetic particles, we replace

$$j(0) = g \left(\varphi^{(-)}(0)\varphi^{(+)}(0) + \frac{\varphi^{(-)2}(0) + \varphi^{(+)2}(0)}{2} \right) \tag{3.10}$$

by

$$j^{\mathrm{s}}(0) = g \varphi^{(-)}(0) \varphi^{(+)}(0) \tag{3.11}$$

in eq. (3.9).

Next we substitute the approximation for $U(0, -\infty)$, cf. eq. (2.21). The operators W_2 and W_3 act trivially on the one-particle state $|\vec{p}\rangle$:

$$W_2 W_3 |\vec{p}\rangle = |\vec{p}\rangle \quad . \tag{3.12}$$

So we are left with

$$G_n(\vec{p}) = g \langle \vec{k}_1, \ldots, \vec{k}_n | \left(\varphi^{(-)}(0)\varphi^{(+)}(0) \right) \mathrm{T} \exp \left\{ -\mathrm{i} \int_{-\infty}^{0} \mathrm{d}t \, H_{\mathrm{I}}^{(+)}(t) \right\} |\vec{p}\rangle \quad . \tag{3.13}$$

Let us remark that we can limit ourselves to the term of order $(n-1)$ in the time ordered expansion in eq. (3.13). Lower orders lead to disconnected graphs which should not contribute to the full amplitude in the asymptotic collinear dynamics considered here. The terms higher than $(n-1)$ necessarily lead to vertex and/or mass corrections neglected throughout this paper.

Therefore

$$\begin{aligned} G_n(\vec{p}) \quad &= \quad g(-\mathrm{i})^{n-1} \int_{-\infty}^{0} \mathrm{d}t_1 \cdots \mathrm{d}t_{n-1} \, \Theta(t_1, \ldots, t_{n-1}) \\ &\quad \times \langle 0 | a(\vec{k}_1) \cdots a(\vec{k}_n) \varphi^{(-)}(0) \varphi^{(+)}(0) H_{\mathrm{I}}^{(+)}(t_1) \cdots H_{\mathrm{I}}^{(+)}(t_{n-1}) |\vec{p}\rangle \quad , \end{aligned} \tag{3.14}$$

where the symbol $\Theta(t_1,\ldots,t_{n-1})$ denotes

$$\Theta(t_1,\ldots,t_{n-1}) = \Theta(t_1-t_2)\cdots\Theta(t_{n-2}-t_{n-1}) \quad . \tag{3.15}$$

It is useful to rewrite G_n as

$$G_n(\vec{p}) = \sum_{r=1}^{n} g_r^n(\vec{p}) \tag{3.16}$$

where

$$\begin{aligned} g_r^n &= g(-\mathrm{i})^{n-1}\int_{-\infty}^{0} \mathrm{d}t_1\cdots\mathrm{d}t_{n-1}\,\Theta(t_1,\ldots,t_{n-1}) \\ &\quad\times\langle 0|a(\vec{k}_1)\cdots a(\vec{k}_{r-1})a(\vec{k}_{r+1})\cdots a(\vec{k}_n)\varphi^{(+)}(0)H_{\mathrm{I}}^{(+)}(t_1)\cdots H_{\mathrm{I}}^{(+)}(t_{n-1})|\vec{p}\rangle \quad . \end{aligned} \tag{3.17}$$

In evaluating the matrix elements in eq. (3.17) we have to commute the $a(\vec{k}_i)$ with $H_{\mathrm{I}}(t_j)$. Thus we shall encounter the following commutators of $a(\vec{k})$:

$$D(t,\vec{k}) = -\mathrm{i}[a(\vec{k}),H_{\mathrm{I}}^{(+)}(t)] = \mathrm{i}g\int_{x^0=t}\mathrm{d}^3x\,\varphi^{(-)}(x)\varphi^{(+)}(x)\mathrm{e}^{\mathrm{i}kx} \quad , \tag{3.18}$$

$$d(t,\vec{k},\vec{k}\,') = [a(\vec{k}),D(t,\vec{k}\,')] = \mathrm{i}g\int_{x^0=t}\mathrm{d}^3x\,\varphi^{(+)}(x)\mathrm{e}^{\mathrm{i}(k+k')x} \quad , \tag{3.19}$$

$$[a(\vec{k}\,''),d(t,\vec{k},\vec{k}\,')] = 0 \quad . \tag{3.20}$$

The operator d is linear in $\varphi^{(+)}$. Therefore we should consider the commutator of d with $H_{\mathrm{I}}^{(+)}$ again. Since d depends on the sum of two intermediate momenta it leads to vertex and mass corrections which—as already stated above—are beyond the scope of our work. As a consequence, d can be treated as a commuting quantity and it is easy to see that for this reason it does not contribute at all. In this way we obtain

$$\begin{aligned} &(-\mathrm{i})^{n-1}\langle 0|a(\vec{k}_1)\cdots a(\vec{k}_{r-1})a(\vec{k}_{r+1})\cdots a(\vec{k}_n)\varphi^{(+)}(0)H_{\mathrm{I}}(t_1)\cdots H_{\mathrm{I}}(t_{n-1})|\vec{p}\rangle = \\ &\quad = \sum_{\pi_r}\langle 0|\varphi^{(+)}(0)D(t_1,\vec{k}_{\pi(1)})\cdots D(t_{n-1},\vec{k}_{\pi(n)})|\vec{p}\rangle \quad , \end{aligned} \tag{3.21}$$

where the summation runs over all permutations of the numbers $1,2,\ldots,r-1,r+1,\ldots,n$.

Applying consecutively

$$D(t,\vec{k})|\vec{p}\rangle = \mathrm{i}g\frac{\mathrm{e}^{\mathrm{i}(E(\vec{p}-\vec{k})+k^0-p^0)t}}{2E(\vec{p}-\vec{k})}|\vec{p}-\vec{k}\rangle \quad , \tag{3.22}$$

and integrating, we arrive at

$$\begin{aligned} &g_r^n(p) = \\ &\quad = g^n\sum_{\pi_r}\frac{1}{2E(\vec{p}-\vec{k}_{\pi(n)})}\times\frac{1}{E(\vec{p}-\vec{k}_{\pi(n)})+k^0_{\pi(n)}-p^0} \\ &\quad\times\frac{1}{2E(\vec{p}-\vec{k}_{\pi(n)}-\vec{k}_{\pi(n-1)})}\times\frac{1}{E(\vec{p}-\vec{k}_{\pi(n)}-\vec{k}_{\pi(n-1)})+k^0_{\pi(n)}+k^0_{\pi(n-1)}-p^0} \\ &\quad\vdots \\ &\quad\times\frac{1}{2E(\vec{p}-\sum_{1,i\neq r}^{n}\vec{k}_{\pi(i)})}\times\frac{1}{E(\vec{p}-\sum_{1,i\neq r}^{n}\vec{k}_{\pi(i)})+\sum_{1,i\neq r}^{n}k^0_{\pi(i)}-p^0} \quad . \end{aligned} \tag{3.23}$$

Having this form of g_r^n, we now look for the kinematical region, that gives a dominant contribution. Note that for massive particles the denominators in g_r^n never vanish.

We shall calculate the dominant contributions in the leading logarithmic approximation, so we consider fixed transverse momenta and

$$\left|k^{\parallel}_{\pi(s)}\right| \gg \sqrt{\vec{k}^{\perp 2}_{\pi(s)} + m^2} \tag{3.24}$$

$$\left|q^{\parallel}_{r,s}\right| \gg \sqrt{\vec{q}^{\perp 2}_{r,s} + m^2} \tag{3.25}$$

for

$$s = 1, 2, \ldots, r-1, r+1, \ldots, n \tag{3.26}$$

and with

$$q_{r,s} = p - \sum_{i=s, i\neq r}^{n} k_{\pi(i)} \quad . \tag{3.27}$$

Then one can show that the terms

$$\frac{1}{2E(\vec{q}_{r,s})[E(\vec{q}_{r,s}) - q^0_{r,s}]} \tag{3.28}$$

in eq. (3.23) are different from zero for

$$k^{\parallel}_{\pi(s)} \gg q^{\parallel}_{r,s} \tag{3.29}$$

or

$$k^{\parallel}_{\pi(s)} = O(q^{\parallel}_{r,s}) \quad . \tag{3.30}$$

The second possibility does not contribute in leading logarithmic order, so we are left with the conditions eq. (3.29). These, together with momentum conservation (see eq. (3.8))

$$k^{\parallel}_r = q^{\parallel}_{r,1} \tag{3.31}$$

lead to strong ordering of the k's

$$p^{\parallel} \gtrsim k^{\parallel}_{\pi(n)} \gg k^{\parallel}_{\pi(n-1)} \gg \ldots \gg k^{\parallel}_{\pi(r+1)} \gg k^{\parallel}_{\pi(r-1)} \gg \ldots \gg k^{\parallel}_{\pi(1)} \gg k^{\parallel}_r \gg m \quad . \tag{3.32}$$

It means that in the expression eq. (3.8) (with $\vec{p}_1^{\perp} = 0$) only the same orderings in both G_n enter, i. e.

$$\begin{aligned} V_n \;=\; & g^{2n} \int \mathrm{d}\tilde{k}_1 \cdots \mathrm{d}\tilde{k}_n \frac{\delta(\vec{p}_1 - \sum_1^n \vec{k}_s)}{{p_1}^0 + {p_2}^0 - \sum_{i=1}^n {k_i}^0} \sum_{r=1}^{n} \sum_{\pi_r} f_r^n(\pi) \\ & \times \frac{1}{m^2 + (\vec{p}_3^{\perp} - \vec{k}^{\perp}_{\pi(n)})^2} \times \frac{1}{m^2 + \vec{k}^{\perp 2}_{\pi(n)}} \cdots \\ & \times \frac{1}{m^2 + (\vec{p}_3^{\perp} - \sum_{1,s\neq r}^n \vec{k}^{\perp}_{\pi(s)})^2} \times \frac{1}{m^2 + (\sum_{1,s\neq r}^n \vec{k}^{\perp}_{\pi(s)})^2} \end{aligned} \tag{3.33}$$

where $f_r^n(\pi)$ is unity in the region defined by eq. (3.32) and zero otherwise.

Integrating over transverse momenta we arrive at

$$V_n = \frac{g^2}{2(2\pi)^3}\left(g^2 K(\vec{p}_3^{\perp 2})\right)^{n-1}\sum_{r=1}^{n}\sum_{\pi_r} I_r^n(\pi) \tag{3.34}$$

where

$$I_r^n(\pi) = \int \frac{\mathrm{d}k_1^\parallel \cdots \mathrm{d}k_n^\parallel}{k_1{}^0 \cdots k_n{}^0} \times \frac{f_r^n(\pi)\delta(p_1^\parallel - \sum_1^n k_i^\parallel)}{p_1{}^0 + p_2{}^0 - \sum_1^n k_i{}^0} \tag{3.35}$$

and

$$K = \frac{1}{2(2\pi)^3}\int \frac{\mathrm{d}^2 k^\perp}{(m^2 + \vec{k}^{\perp 2})(m^2 + (\vec{p}_3^\perp - \vec{k}^\perp)^2)} \quad . \tag{3.36}$$

The longitudinal integration yields the same result for each I_r^n:

$$I_r^n(\pi) = \frac{2}{s}\frac{(\ln s)^{n-1}}{(n-1)!}\left(1 + O\left(\frac{1}{\ln s}\right)\right) \tag{3.37}$$

where $s \approx 2mp_1^\parallel$, so that

$$V_n = n!\frac{g^2}{(2\pi)^3}\frac{(g^2 K \ln s)^{n-1}}{(n-1)!}\frac{1}{s} \quad . \tag{3.38}$$

Then using eq. (3.7) results in

$$\mathcal{M}_2 = \frac{g^2}{s}\sum_{n=1}^{\infty}\frac{(g^2 K \ln s)^{n-1}}{(n-1)!} = g^2 s^{-1+g^2 K} \quad , \tag{3.39}$$

i. e. we have obtained Regge behaviour [9].

4 Eikonalization

In this chapter we apply the factorization formula eq. (2.21) to derive the familiar eikonal form of the scattering amplitude.

We start with

$$\langle \vec{p}_3, \vec{p}_4 | S | \vec{p}_1, \vec{p}_2 \rangle \quad , \quad S = U(\infty, -\infty) = W_1(\infty, -\infty) W_2(\infty, -\infty) W_3(\infty, -\infty) \; . \tag{4.1}$$

In this matrix element the part $W_1 W_2$ can be replaced by unity; other terms of $W_1 W_2$ either give a one-particle s-channel contribution that can be neglected, or vanish identically, because W_3 conserves the particle number. The first term in the exponent of W_3, eq. (2.23), is also non-leading for large c. m. s. energy.

In this way we arrive at

$$\langle \vec{p}_3, \vec{p}_4 | S | \vec{p}_1, \vec{p}_2 \rangle = \langle \vec{p}_3, \vec{p}_4 | \mathrm{T}\exp\left\{ \int_{-\infty}^{\infty} r(t)\,\mathrm{d}t + O(g^3) \right\} | \vec{p}_1, \vec{p}_2 \rangle \quad , \tag{4.2}$$

where

$$r(t) = -g^2 \int_{-\infty}^{t} \mathrm{d}y^0 \int_{x^0 = t} \mathrm{d}^3 x\, \mathrm{d}^3 y\, \varphi^{(-)}(x)\varphi^{(-)}(y)\varphi^{(+)}(x)\varphi^{(+)}(y)\Delta^+(x-y) \quad . \tag{4.3}$$

In fact, for large energies the time ordering in eq. (4.2) is irrelevant. To see this, consider any two-particle state $|\vec{k}_1, \vec{k}_2\rangle$ in the center of mass system of k_1 and k_2. Then we have

$$
\begin{aligned}
r(t)r(t')|\vec{k}_1, -\vec{k}_1\rangle \quad = \quad & -\frac{g^4}{4}\int \mathrm{d}\tilde{q}\,\mathrm{d}\tilde{q}'\, a^\dagger(\vec{k}_1+\vec{q}+\vec{q}')a^\dagger(-\vec{k}_1-\vec{q}-\vec{q}')|0\rangle \\
& \times\frac{\mathrm{e}^{-2\mathrm{i}({k_1}^0-E(\vec{k}_1+\vec{q}))t}}{(E(\vec{k}_1+\vec{q}))^2[{k_1}^0-q^0-E(\vec{k}_1+\vec{q})+\mathrm{i}\varepsilon]} \\
& \times\frac{\mathrm{e}^{-2\mathrm{i}(E(\vec{k}_1+\vec{q})-E(\vec{k}_1+\vec{q}+\vec{q}'))t'}}{(E(\vec{k}_1+\vec{q}+\vec{q}'))^2[E(\vec{k}_1+\vec{q})-q'^0-E(\vec{k}_1+\vec{q}+\vec{q}')+\mathrm{i}\varepsilon]} \quad .
\end{aligned}
\tag{4.4}
$$

In the limit ${k_1}^0 \to \infty$, eq. (4.4) reduces to

$$
\begin{aligned}
r(t)r(t')|\vec{k}_1, -\vec{k}_1\rangle \quad \stackrel{{k_1}^0\to\infty}{=} \quad & -\frac{g^4}{4({k_1}^0)^4}\int \mathrm{d}\tilde{q}\,\mathrm{d}\tilde{q}'\, a^\dagger(\vec{k}_1+\vec{q}+\vec{q}')a^\dagger(-\vec{k}_1-\vec{q}-\vec{q}')|0\rangle \\
& \times\frac{\mathrm{e}^{2\mathrm{i}q^\parallel t}}{q^0+q^\parallel-\mathrm{i}\varepsilon}\times\frac{\mathrm{e}^{2\mathrm{i}q'^\parallel t'}}{q'^0+q'^\parallel-\mathrm{i}\varepsilon} \quad ,
\end{aligned}
\tag{4.5}
$$

from which it is evident that

$$
[r(t), r(t')]|\vec{k}_1, -\vec{k}_1\rangle \stackrel{{k_1}^0\to\infty}{=} 0 \quad . \tag{4.6}
$$

As a consequence, eq. (4.2) can be written as

$$
\langle\vec{p}_3, \vec{p}_4|S|\vec{p}_1, \vec{p}_2\rangle = \langle\vec{p}_3, \vec{p}_4|\exp\left\{\int_{-\infty}^{\infty} r(t)\,\mathrm{d}t\right\}|\vec{p}_1, \vec{p}_2\rangle \tag{4.7}
$$

with

$$
\int_{-\infty}^{\infty} r(t)\,\mathrm{d}t\,|\vec{p}_1, \vec{p}_2\rangle = \mathrm{i}\frac{g^2}{8(2\pi)^2({p_1}^0)^2}\int\frac{\mathrm{d}^2q^\perp}{\vec{q}^{\perp 2}+m^2}a^\dagger(\vec{p}_1+\vec{q}^\perp)a^\dagger(\vec{p}_2-\vec{q}^\perp)|0\rangle \quad . \tag{4.8}
$$

From eq. (4.8) it is straightforward to derive the well-known eikonal formula

$$
\mathcal{M}_{\mathrm{fi}} = 2\mathrm{i}s\int \mathrm{d}^2b\,\mathrm{e}^{\mathrm{i}\vec{p}_3^\perp\vec{b}}\left[\exp\left(\frac{\mathrm{i}g^2}{4\pi s}K_0(m|\vec{b}|)\right)-1\right] \quad . \tag{4.9}
$$

5 Concluding remarks

We have shown how the collinear asymptotic dynamics described by the Hamiltonian eq. (2.5) restricted to the kinematical region Ω, eq. (2.9), leads to both Reggeization and eikonalization of two-particle scattering. While Reggeization is dominated by three-particle collinear dynamics, eikonalization corresponds to that kinematical region of the asymptotic Hamiltonian where only two particles are collinear. In contradistinction to infrared dynamics [3, 6], the time ordering of the dressing operator is essential for three-particle collinear dynamics which leads to Reggeization.

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